Last time: linear function $W \xrightarrow{f} V$; two equivalent ways to find the matrix $A \in \mathbb{R}^{m\times n}$ corresponding to f

basis $\underline{w} = \{w_1, \dots, w_n\}$ (of W); basis $\underline{v} = \{v_1, \dots, v_m\}$ (of V)

1) write $f(w_i) = a_{1i} v_1 + \dots + a_{mi} v_m \quad \forall\, 1 \le i \le n$

and construct
$$A = \begin{pmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mi} & \cdots & a_{mn} \end{pmatrix} \in \mathbb{R}^{m\times n}$$

2) Φ:
$$\begin{tikzcd} \mathbb{R}^n \arrow[r, bend left, "h^{-1}"] & W \arrow[l, "\cong"', "h"] \arrow[r, "f"] & V \arrow[r, "\cong", "g"'] & \mathbb{R}^m \end{tikzcd}$$

where $g : V \xrightarrow{\sim} \mathbb{R}^m$, $g(z) = [z]_{\underline{v}}$, $\forall z \in V$

$h : W \xrightarrow{\sim} \mathbb{R}^n$, $h(z) = [z]_{\underline{w}}$, $\forall z \in W$

since $\Phi = g \circ f \circ h^{-1} : \mathbb{R}^n \to \mathbb{R}^m$ is linear,

it corresponds to a matrix $A \in \mathbb{R}^{m\times n}$, i.e. $\Phi(x) = Ax$

Also, remember that if $V = \mathbb{R}^m$ with basis $\underline{v} = \{v_1, \dots, v_m\}$

$W = \mathbb{R}^n$ with basis $\underline{w} = \{w_1, \dots, w_n\}$

the isomorphism $g : \mathbb{R}^m \to \mathbb{R}^m$ is given by $g(x) = P_{\underline{v}}^{-1} x$

$h : \mathbb{R}^n \to \mathbb{R}^n$ is given by $h(x) = P_{\underline{w}}^{-1} x$

where $P_{\underline{v}} = P_{\underline{e} \leftarrow \underline{v}}$ is the $m \times m$ matrix with columns $v_1, \dots, v_m$

$P_{\underline{w}} = P_{\underline{e} \leftarrow \underline{w}}$ is the $n \times n$ matrix with columns $w_1, \dots, w_n$

Today : let us combine these points of view ; suppose

$f : \mathbb{R}^n \to \mathbb{R}^m$ is given by the matrix

$$B \in \mathbb{R}^{m \times n} \iff f(x) = Bx$$

$$\Phi : \mathbb{R}^n \xrightarrow{h^{-1}} \mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}^m$$ is given $\left(\Phi(x) = Ax \right)$ by the matrix

$$\boxed{A = P_{\underline{v}}^{-1} B P_{\underline{w}}}$$

(this is called the change of basis formula for matrices)

In other words, $A = P_{\underline{v} \leftarrow \underline{e}} B P_{\underline{e} \leftarrow \underline{w}}$, so

this amounts to "changing B from the $\underline{e}$-$\underline{e}$ basis to A in the $\underline{v}$-$\underline{w}$ basis"

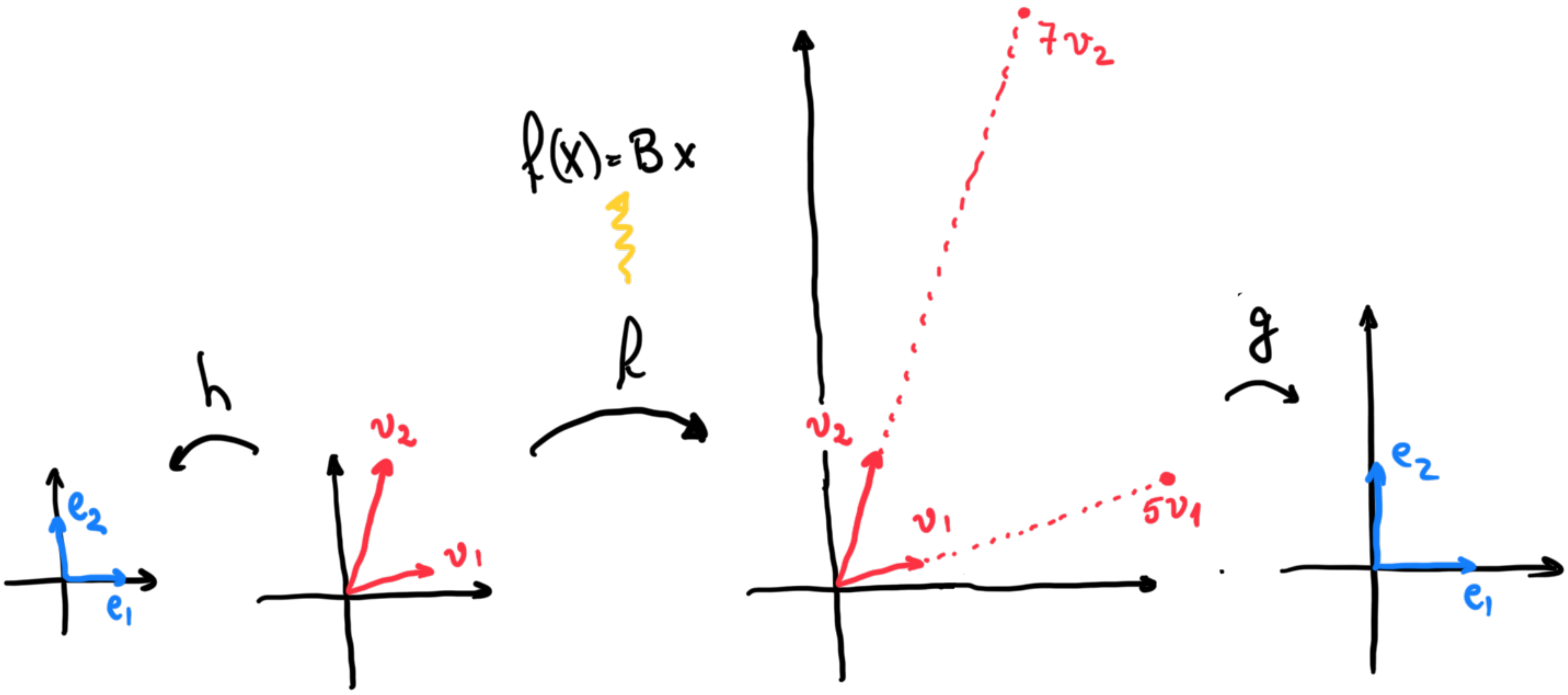
$$A[\;] = P \; B \, P \; [z] = P \; Bz = [Bz]$$

i.e. $A[z]_{w} = P_{v \leftarrow e} B P_{e \leftarrow w} [z]_{w} = P_{v \leftarrow e} B z = [Bz]_{v}$

$\Downarrow$

THM 16.1: $A[z]_{w} = [Bz]_{v}, \quad \forall z \in \mathbb{R}^n$

Ex: find the matrix which represents $f : \mathbb{R}^2 \to \mathbb{R}^2$ which a dilation by a factor of 5 in the v_1 direction and a factor of 7 in the v_2 direction where $v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$



above, we define $g = h : \mathbb{R}^2 \to \mathbb{R}^2$ which take v_1 to e_1, v_2 to e_2

$\Downarrow$

$\Phi = g \circ f \circ h^{-1}$ will take $e_1 \rightsquigarrow v_1 \rightsquigarrow 5v_1 \rightsquigarrow 5e_1$

$e_2 \rightsquigarrow v_2 \rightsquigarrow 7v_2 \rightsquigarrow 7e_2$

$\Downarrow$

$\Phi x = Ax$, where $A = \begin{pmatrix} 5 & 0 \\ 0 & 7 \end{pmatrix} = P_{v \leftarrow e} \cdot B \cdot P_{e \leftarrow v}$

But $P_{\underline{e} \leftarrow \underline{v}} = \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$

$$P_{\underline{v} \leftarrow \underline{e}} = \left(P_{\underline{e} \leftarrow \underline{v}}\right)^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}^{-1} = \frac{1}{5}\begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$$

Interlude: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\underbrace{ad-bc}_{\det}}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Proof: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} ad-bc & -ab+ba \\ cd-dc & -bc+ad \end{pmatrix} = I_2$

Solve for B: $\begin{pmatrix} 5 & 0 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}^{-1} B \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$

$$\Downarrow$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 7 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}^{-1} = B$$

$$\Downarrow$$

$$\begin{pmatrix} 10 & 7 \\ 5 & 21 \end{pmatrix} \cdot \frac{1}{5}\begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} = B \implies B = \frac{1}{5}\begin{pmatrix} 23 & 4 \\ -6 & 37 \end{pmatrix}$$

New old topic: any vector space V of dimension n is $V \cong \mathbb{R}^n$

isomorphic, i.e $\exists$ linear bijection between them

COR 16.2: any two vector spaces V and W of the same dimension are isomorphic: $V \cong W$

dimension are isomorphic ...

$\Downarrow$

the dimension is the most important property of a vector space

New old topic: vector spaces arising from a matrix $A \in \mathbb{R}^{m \times n}$

- $\mathrm{Ker}(A) = \mathrm{Null}(A) \subseteq \mathbb{R}^n$
- $\mathrm{Im}(A) = \mathrm{Col}(A) \subseteq \mathbb{R}^m$
- $\mathrm{Row}(A) = \mathrm{Col}(A^T) \subseteq \mathbb{R}^n$

$$A \rightsquigarrow \mathrm{REF}(A) = \tilde{A} = \begin{pmatrix} \boxed{1} & \ast & 0 & \ast & \ast & 0 & \ast & \ast & \ast \\ 0 & \ast & \boxed{1} & \ast & \ast & 0 & \ast & \ast & \ast \\ 0 & \ast & 0 & \ast & \ast & \boxed{1} & \ast & \ast & \ast \end{pmatrix}$$

Note: if you only need to calculate the number of pivots, any echelon form of A will do (you don't need to go all the way to the reduced echelon form)

Let $K = \#$ of pivots of $\tilde{A} \rightsquigarrow 0 \leq K \leq \min(m,n)$

Fact: $\mathrm{Ker}(A) = \mathrm{Ker}(\tilde{A})$

dim = # of free columns

= # all columns − # pivot columns

= $n - k$

Fact: $\mathrm{Col}(A) \cong \mathrm{Col}(\tilde{A})$

same dimension $= k$

spanned by pivot columns of A

spanned by pivot columns of $\tilde{A}$

DEF 16.3: $\forall$ matrix $A \in \mathbb{R}^{m \times n}$

$$\mathrm{rank}(A) := \dim \mathrm{Col}(A)$$

$$\mathrm{nullity}(A) := \dim \mathrm{Ker}(A)$$

THM 16.4 (rank-nullity theorem): $\forall A \in \mathbb{R}^{m \times n}$

$$n = \mathrm{rank}(A) + \mathrm{nullity}(A)$$

$= k$ $= n - k$

[illegible]   (# free columns of $\tilde{A}$)

(# pivots of A) (# free columns of A)

THM 16.5: $0 \leq \text{rank}(A) \leq \min(m,n)$

$$\max(0, n-m) \leq \text{nullity}(A) \leq n$$

$$n = n - 0 \geq \text{nullity}(A) = n - \text{rank}(A) \geq n - \min(m,n) = \max(0, n-m)$$

Ex: $A = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} \in \mathbb{R}^{3\times 5}$

$$0 \leq \text{rank}(A) = \dim \text{Col}(A) \leq 3$$

$$2 \leq \text{nullity}(A) = \dim \text{Ker}(A) \leq 5$$

THM 16.6: $\forall\, r \in \{0, \ldots, \min(m,n)\}$,

$\exists$ a matrix $A \in \mathbb{R}^{m\times n}$ of rank r

$$\|$$

$$\begin{pmatrix} \overbrace{\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix}}^{r \text{ ones}} & 0 \\ & \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \end{pmatrix}$$

(the only matrix of rank 0 is the all-zero matrix)

Square matrices: $m = n$

$$\Downarrow$$

$$\operatorname{rank}(A \in \mathbb{R}^{n\times n}) \in \{0, 1, \ldots, n\}$$

THM 16.7: for any square matrix $A \in \mathbb{R}^{n\times n}$,

$$\begin{tikzcd}
\operatorname{rank}(A) = n \arrow[r, Leftrightarrow] \arrow[d, Leftrightarrow] & A \text{ invertible} \arrow[d, Leftrightarrow] \\
\operatorname{Ker}(A) = \{0\} \arrow[dr, Leftrightarrow] & \det(A) \neq 0 \arrow[d, Rightarrow] \\
& \text{columns of } A \text{ form a basis of } \mathbb{R}^n
\end{tikzcd}$$

New topic: any rank r matrix $B \in \mathbb{R}^{m\times n}$

is$^{(*)}$

$$A = \begin{array}{c} n \text{ columns} \\ \begin{pmatrix} \underbrace{\begin{matrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{matrix}}_{r \text{ ones}} & 0 \\ 0 & 0 \end{pmatrix} \end{array} \; m \text{ rows}$$

"when expressed in suitable bases"

THM 16.8: for any $B \in \mathbb{R}^{m\times n}$ of rank r, there exist invertible matrices $P \in \mathbb{R}^{m\times m}$ and $Q \in \mathbb{R}^{n\times n}$ such that

$$B = P \begin{pmatrix} \begin{matrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{matrix} & 0 \\ \underbrace{\quad 0 \quad}_{r \text{ ones}} & 0 \end{pmatrix} Q$$

Before proving this theorem, let's generalize the setup

$$f : \underset{\text{dim } n}{W} \longrightarrow \underset{\text{dim } m}{V}$$

$$\mathrm{Ker}(f) = \{ w \in W \mid f(w) = 0 \} \subseteq W$$

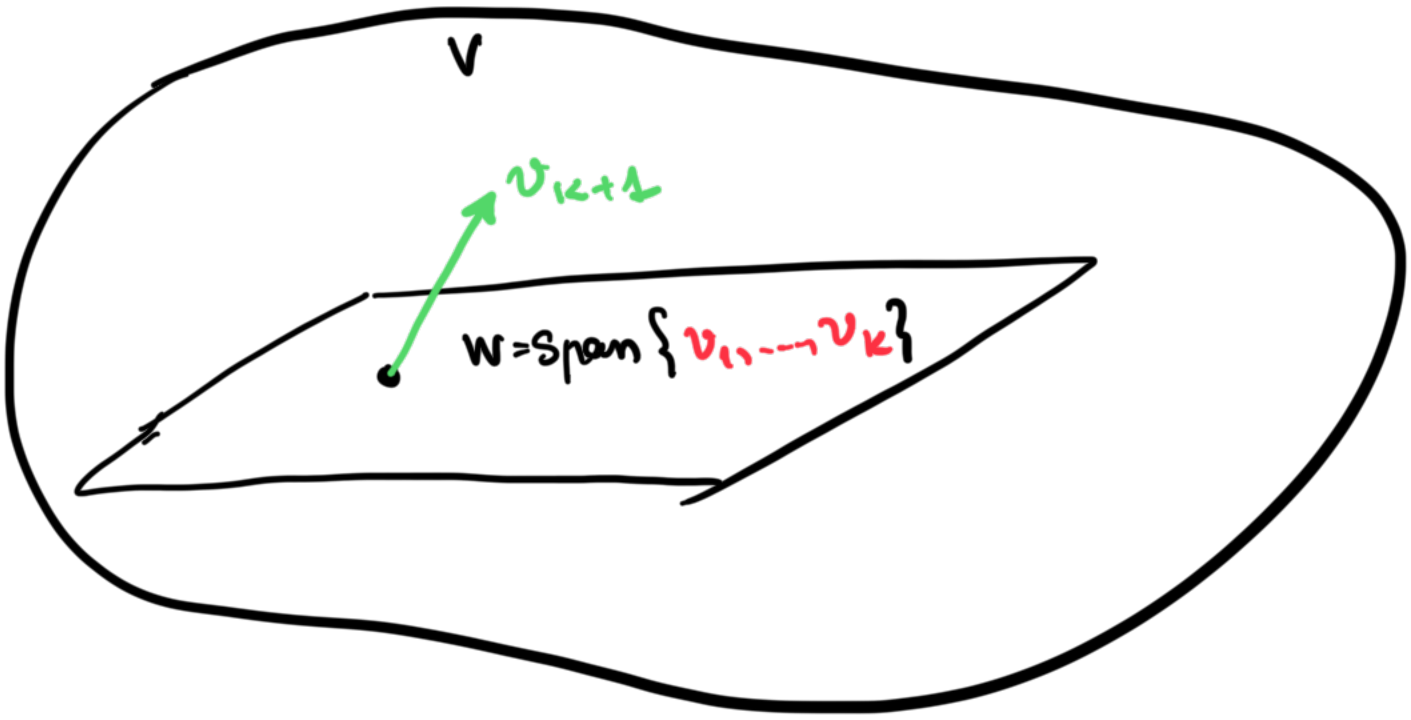
$$\mathrm{Im}(f) = \{ v \in V \mid \exists\, w \text{ s.t. } f(w) = v \} \subseteq V$$

- $\mathrm{rank}(f) := \dim \mathrm{Im}(f)$
- $\mathrm{nullity}(f) := \dim \mathrm{Ker}(f)$

Rank-nullity theorem for f

$\mathrm{rank}(f) + \mathrm{nullity}(f) = \dim(W)$

PROP: $\forall$ vector space V of dimension m suppose you have linearly independent vectors $v_1, \dots, v_k \in V$; then we can find $v_{k+1}, \dots, v_m \in V$ s.t. $\{v_1, \dots, v_k, v_{k+1}, \dots, v_m\}$ is a basis of V



step 1) pick $v_{k+1} \notin \text{span}\{v_1, \dots, v_k\}$

- is $V = \text{span}\{v_1, \dots, v_k, v_{k+1}\}$? if Yes, we are done

if No, go to step 2

step 2) pick $v_{k+2} \notin \text{span}\{v_1, \dots, v_k, v_{k+1}\}$

- is $V = \text{span}\{v_1, \dots, v_k, v_{k+1}, v_{k+2}\}$? if Yes, Done.

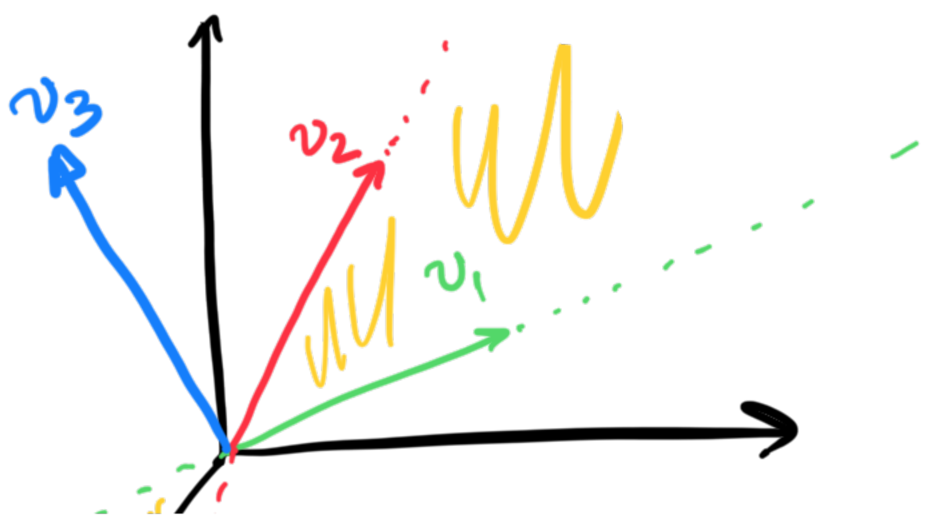
if No,

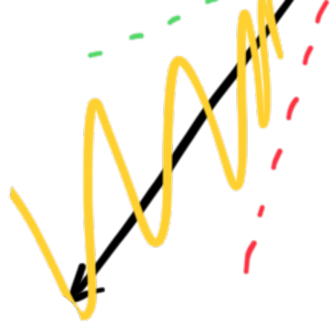
$\vdots$

m−k) pick $v_m \notin \text{span}\{v_1, \dots, v_k, v_{k+1}, v_{k+2}, \dots, v_{m-1}\}$

Then $v_1, \dots, v_k, v_{k+1}, v_{k+2}, \dots, v_{m-1}, v_m$ are linearly independent, and so they form a basis of the m-dimensional space V

Example of above setup for $k=1, m=3$





(fact: in a vector space V of dimension n,

vectors $\{v_1, \dots, v_n\}$ are linearly independent

$\Updownarrow$

vectors $\{v_1, \dots, v_n\}$ span V

$\Updownarrow$

vectors $\{v_1, \dots, v_n\}$ form a basis of V.)

Consider any $f : W \longrightarrow V$ of rank r

$\Downarrow$

$\mathrm{Im}(f)$ is a r-dim subspace of V

Write $\mathrm{Im}(f) = \mathrm{Span}\{v_1, \dots, v_r\}$

Pick $v_{r+1}, \dots, v_m \in V$ s.t.

$\underline{v} = \{v_1, \dots, v_m\}$ is a basis of V

$\Leftarrow$

Pick $w_1, \dots, w_r \in W$ s.t. $f(w_1) = v_1, \dots, f(w_r) = v_r$

Claim: $w_1, \dots, w_r$ are linearly independent

(indeed, if $0 = \lambda_1 w_1 + \dots + \lambda_r w_r$ for $\lambda_1, \dots, \lambda_r \in \mathbb{R}$ not all 0, then $0 = \lambda_1 f(w_1) + \dots + \lambda_r f(w_r) = \lambda_1 v_1 + \dots + \lambda_r v_r$, which would contradict the assumption that $v_1, \dots, v_r$ form a basis of $\mathrm{Im}(f)$)

[illegible] to be a basis of $\mathrm{Ker}(f)$

Then pick $w_{r+1}, \dots, w_n$ to be a basis of $\ker(f)$

$\Downarrow$

$\underline{w} = \{w_1, \dots, w_n\}$ forms a basis of W (because they are linearly independent and there are exactly $n = \dim(W)$ of them)

We conclude $f(w_1) = v_1$, where $g : V \xrightarrow{\sim} \mathbb{R}^m$

$v_i \rightsquigarrow e_i$

$$\vdots$$

$$f(w_r) = v_r$$

$h : W \xrightarrow{\sim} \mathbb{R}^n$

$w_i \rightsquigarrow e_i$

$$f(w_{r+1}) = 0$$

$$\vdots$$

$$f(w_n) = 0$$

Now let's conclude the proof of of Theorem 16.8; let

$W = \mathbb{R}^n$ with basis $\underline{w} = \{w_1, \dots, w_n\}$

$V = \mathbb{R}^m$ with basis $\underline{v} = \{v_1, \dots, v_m\}$

$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $f(x) = Bx$ for an arbitrary $B \in \mathbb{R}^{m \times n}$

$g : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is given by $g(x) = P_{\underline{v}}^{-1} x$

$h : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is given by $h(x) = P_{\underline{w}}^{-1} x$

$$\begin{array}{ccccccc}
\Phi : \mathbb{R}^n & \xleftarrow{h} & W & \xrightarrow{f} & V & \xrightarrow{g} & \mathbb{R}^m \\
e_1 & \rightsquigarrow & w_1 & \rightsquigarrow & v_1 & \rightsquigarrow & e_1 \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

$$e_r \rightsquigarrow w_r \rightsquigarrow v_r \rightsquigarrow e_r$$
$$e_{r+1} \rightsquigarrow w_{r+1} \rightsquigarrow 0 \rightsquigarrow 0$$
$$\vdots$$
$$e_n \rightsquigarrow w_n \rightsquigarrow 0 \rightsquigarrow 0$$

It is easy to see that the above composition Φ is

$$\Phi(x) = Ax \text{, where } A = \left(\begin{array}{cc} \underbrace{\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix}}_{r \text{ ones}} & 0 \\ 0 & 0 \end{array}\right)$$

But the change of basis formula for matrices (yellow box at the beginning of this class) gives us

$$A = P_{\underline{v}}^{-1} B P_{\underline{w}} \implies B = P_{\underline{v}} A P_{\underline{w}}^{-1}$$

If we let $P = P_{\underline{v}}$ and $Q = P_{\underline{w}}^{-1}$, then we are done proving Thm 16.8